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1. Introduction

The hydrodynamic theory of jets attempts to determine the surfaces of a jet and the detailed velocity distribution within a jet. The main difficulty stems from the nonlinear boundary conditions to be satisfied on the unknown jet surfaces. When the flow is restricted to be two-dimensional and steady, and surface tension is neglected, this difficulty can be overcome by the method of conformal mapping and the surfaces and flow can be found exactly. This determination is based on the exact equations of hydrodynamics, omitting viscosity.

In order to treat more general problems (unsteady, three-dimensional, or including gravity) the simpler hydraulic theory is usually employed. In this theory one gives up the attempt to find the detailed velocity distribution within the jet, but instead assumes that the velocity is constant on each cross-section of the jet. Since this assumption is incompatible with the exact equations of hydrodynamics, it is necessary to use different, approximate, equations based on the conservation laws of mechanics. These equations can be solved and yield the shape, thickness and speed of the jet, approximately. (This method can also be extended to take account of surface tension, as will be shown below.)

Two questions which immediately arise are: "What is the relationship between the two theories?" and "How can the results of the hydraulic theory be improved?" In this paper we answer these questions by presenting a method of solution of the hydrodynamic problem as a series in powers of the jet thickness divided by some other typical length of the jet. The first term in this solution is found to be the solution given by the hydraulic theory, thus answering the first question. The higher order terms in the series yield corrections to the hydraulic theory, thus answering the second question.

We were led to the solution by noting that in the problem of water waves, which is mathematically similar to that of a jet, both gravity and time dependence are taken into account. Because of the similarity of the two problems, we expected that the methods used in treating water waves could also be applied to jets. We found that this was indeed the case. In fact precisely the same kind of problem arises in water wave theory in showing the connection between the exact theory and the so-called non-linear shallow water theory. (The shallow water theory is the hydraulic theory of water waves.)

The method which we employ involves the introduction of dimensionless variables in such a way that horizontal and vertical distances are stretched by different amounts, just as in the treatment of waves in shallow water. However, we must take account of the fact that the flow is not essentially horizontal, as in the shallow water case, but is parallel to some unknown curve. (In flow over a spillway, which we also consider, this curve is the bottom and is therefore known.) This difference is taken account of in the choice of dimensionless variables. Once appropriate variables have been introduced, the solution is expressed as a series in the ratio of the horizontal to the vertical scale factors. The successive terms in the series are then obtained in a straightforward manner.

This procedure is carried out for two dimensional jets. The results apply to both steady and unsteady jets, with or without gravity. Thus the initial width, speed and direction of the jet may vary in time. In the resulting zero-order or hydraulic theory the individual "particles" of fluid move along parabolic trajectories as if they were freely falling, and the pressure throughout the jet is the same as the external pressure. In the case of flow along a spillway or hill, the flow may detach from the bottom if the bottom curvature is too great.

In section 2 the hydrodynamic problem is formulated, and the dimensionless variables and series expansion are introduced. In section 3 the zero order solution for both steady and unsteady jets is given, and found to agree with the hydraulic theory. In section 4 the first order solution for steady jets is obtained, which represents an improvement over the hydraulic theory. In section 5 the steady and unsteady flow over a rigid surface, such as a hill or spillway, is treated. In section 6 the hydrodynamic formulation is modified to include surface tension, and the resulting zero order equations are found to coincide with those of a hydraulic theory which includes surface tension. The zero order solution for a steady jet is also given. (Appendix I treats vertical steady jets, which require a separate analysis, and both zero and first order solutions are obtained. A very slight modification of this method enables us to treat vertical jets of circular cross section, but this is not given here.)

2. Formulation of the Problem

We consider the two-dimensional, irrotational flow of an incompressible, inviscid fluid bounded by two free streamlines $\bar{y} = \bar{y}_i(\bar{x},\bar{t})$; $i = 1,2$. This flow may be called an unsteady jet. The coordinate system is so chosen that gravity acts in the negative $\bar{y}$ direction. We assume that the pressure has the same constant value on both free streamlines, and this constant will be taken to be zero, without loss of generality. The horizontal velocity $\bar{u}$, the vertical velocity $\bar{v}$, and the pressure $\bar{p}$ satisfy the following differential equations and boundary conditions [1] (ρ is the density and g the acceleration of gravity):

$$\bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} = 0$$

$$\bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} = - \bar{p}_{\bar{x}}/\rho$$

$$\bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} = - \bar{p}_{\bar{y}}/\rho - g \tag{2.1}$$

$$\bar{v}_{\bar{x}} - \bar{u}_{\bar{y}} = 0$$

$$\bar{p} = 0 \quad \text{at} \quad \bar{y} = \bar{y}_i(\bar{x},\bar{t}) \qquad (i = 1,2)$$

$$\bar{y}_{i\bar{t}} + \bar{y}_{i\bar{x}}\,\bar{u} = \bar{v} \quad \text{at} \quad \bar{y} = \bar{y}_i(\bar{x},\bar{t}) \qquad (i = 1,2) \quad .$$

In addition to these equations, the values of $\bar{u}$, $\bar{v}$, $\bar{p}$, $\bar{y}_i$ must be given at some cross-section of the jet (the "nozzle"). These conditions will be considered after we obtain a general expression for the solution.

We now introduce dimensionless variables, choosing two constants a and b with dimensions of length, as scale factors in the horizontal and vertical directions respectively. These constants may be typical dimensions of the jet in the respective directions. The dimensionless variables are defined by:

$$\bar{x} = ax \qquad \bar{t} = \sqrt{\frac{a}{g}}\, t \qquad \bar{p} = ag\rho p$$

$$\bar{y} = ay_o(x,t) + by \qquad \bar{u} = \sqrt{ag}\, u \tag{2.2}$$

$$\bar{y}_i(\bar{x}) = ay_o(x,t) + by_i(x,t) \qquad \bar{v} = \sqrt{ag}\, v \quad .$$

In these equations $\bar{y} = ay_o(x,t)$ is the equation of the as yet undetermined center line of the jet, i.e. $y_o(x,t)$ is to be chosen such that $\bar{y}_1 + \bar{y}_2 = 2ay_o$. From this it follows that

$$y_1(x,t) = - y_2(x,t) \quad . \tag{2.3}$$

Inserting the new variables, equations (2.2), into equations (2.1) we obtain (letting $\frac{b}{a} = \sigma$)

$$\sigma u_x - y_{ox}u_y + v_y = 0$$

$$\sigma u_t - y_{ot}u_y + \sigma uu_x - y_{ox}uu_y + vu_y = -\sigma p_x + y_{ox}p_y$$

$$\sigma v_t - y_{ot}v_y + \sigma uv_x - y_{ox}uv_y + vv_y = - p_y - \sigma$$

$$\sigma v_x - y_{ox}v_y - u_y = 0 \tag{2.4}$$

$$p = 0 \quad \text{at} \quad y = y_i(x,t)$$

$$y_{ot} + \sigma y_{ox} + uy_{ox} + \sigma uy_{ix} = v \quad \text{at} \quad y = y_i(x,t) \quad .$$

The parameter σ in equations (2.4) will be small if the vertical dimension b is small compared to the horizontal dimension a, which is the case for thin jets. In such cases we may seek a solution of equations (2.4) in the form of a power series in σ . Thus we let

$$u = \sum_{n=0}^{\infty} u^{(n)}(x,y,t)\sigma^n \quad , \quad v = \sum_{n=0}^{\infty} v^{(n)}(x,y,t)\sigma^n \quad ,$$

(2.5)

$$p = \sum_{n=0}^{\infty} p^{(n)}(x,y,t)\sigma^n \quad , \quad y_i = \sum_{n=0}^{\infty} y_i^{(n)}(x,y,t)\sigma^n \quad .$$

To determine the coefficients in these expansions we insert equations (2.5) into equations (2.3) and (2.4) and equate the coefficients of each power of σ to zero. In this way we obtain from the coefficients of σ^0:

(2.6)

a) $$y_1^{(o)}(x,t) = -y_2^{(o)}(x,t)$$

b) $$-y_{ox}u_y^{(o)} + v_y^{(o)} = 0$$

c) $$-y_{ot}u_y^{(o)} - y_{ox}u^{(o)}u_y^{(o)} + v^{(o)}u_y^{(o)} = y_{ox}p_y^{(o)}$$

d) $$-y_{ot}v_y^{(o)} - y_{ox}u^{(o)}v_y^{(o)} + v^{(o)}v_y^{(o)} = -p_y^{(o)}$$

e) $$y_{ox}v_y^{(o)} + u_y^{(o)} = 0$$

f) $$p^{(o)} = 0 \quad \text{at} \quad y = y_i^{(o)}(x,t)$$

g) $$y_{ot} + u^{(o)}y_{ox} = v^{(o)} \quad \text{at} \quad y = y_i^{(o)}(x,t).$$

The procedure thus far is nearly the same as that of [2] and [3]. The essential difference occurs in the choice of dimensionless variables.

3. Zero-Order Solution for Unsteady Jets

From equations (2.6b) and (2.6e) it follows that

(3.1) $$u^{(o)} = u^{(o)}(x,t) \quad , \quad v^{(o)} = v^{(o)}(x,t) \quad .$$

Now using equations (3.1) in (2.6c), we have $p_y^{(o)} = 0$, and then from (2.6f)

(3.2) $$p^{(o)} = 0 \quad .$$

With these results, equations (2.6b)-(2.6f) are identically satisfied, and only equations (2.6a) and (2.6g) remain. These equations are insufficient to complete the determination of the zero order terms, and therefore we proceed to equate to zero the coefficients of σ^1 in equations (2.3) and (2.4). We obtain:

$$(3.3) \quad a) \qquad y_1^{(1)}(x,t) = -y_2^{(1)}(x,t)$$

$$b) \qquad u_x^{(o)} - y_{ox}u_y^{(1)} + v_y^{(1)} = 0$$

$$c) \qquad u_t^{(o)} + u^{(o)}u_x^{(o)} = y_{ox}p_y^{(1)}$$

$$d) \qquad v_t^{(o)} + u^{(o)}v_x^{(o)} = -p_y^{(1)} - 1$$

$$e) \qquad v_x^{(o)} - y_{ox}v_y^{(1)} - u_y^{(1)} = 0$$

$$f) \qquad p^{(1)} = 0 \quad \text{at} \quad y = y_i^{(o)}$$

$$g) \qquad y_{it}^{(o)} + u^{(o)}y_{ix}^{(o)} + u^{(1)}y_{ox} = v^{(1)} \quad \text{at} \quad y = y_i^{(o)}$$

From equations (3.3c) and (3.1) we find that $p_y^{(1)}$ is independent of y. Since $p^{(1)}$ is zero at two different values of y, by equation (3.3f), we have

$$(3.4) \qquad p^{(1)} = 0 \quad .$$

Then equation (3.3c) becomes

$$(3.5) \qquad u_t^{(o)} + u^{(o)}u_x^{(o)} = 0 \quad .$$

This last equation implies that the zero-order horizontal velocity component is a constant for each "particle" of the fluid. This suggests the following method for obtaining $u^{(o)}$. Let

$$(3.6) \qquad u^{(o)} = A(\tau)$$

where A is an arbitrary function of a parameter τ. This parameter τ is defined implicitly in terms of x and t by the equation

$$(3.7) \qquad x = (t-\tau)A(\tau) + x^*(\tau) \quad .$$

Here $x^*(\tau)$ is also an arbitrary function. We may interpret τ as the time at which the particle at x and time t was at x^*. For example, $x^*(\tau)$ might be the x-coordinate of the nozzle at time τ, and the emitted particles would then have the velocity $A(\tau)$. It is easily verified by differentiations that equation (3.6) gives a solution of equation (3.5), provided τ is defined by equation (3.7).

In the same way, a solution of equation (3.3d) for $v^{(o)}(x,t)$ is

$$v^{(o)} = -(t-\tau) + W(\tau) \tag{3.8}$$

where $W(\tau)$ is an arbitrary function. Now we may solve equation (2.6g) for $y_o(x,t)$ by assuming that $y_o(x,t) \equiv B(\tau,t)$ i.e. y_o is a function of τ and t. Then equation (2.6g) becomes

$$B_t = v^{(o)} \quad . \tag{3.9}$$

Using equation (3.8) in equation (3.9) and integrating we have for B or y_o

$$y_o(x,t) = y^*(\tau) + (t-\tau)\,W(\tau) - \frac{(t-\tau)^2}{2} \quad . \tag{3.10}$$

Here $y^*(\tau)$ is an arbitrary function which can be interpreted as the y-coordinate of the nozzle at time τ.

Before considering $y_1^{(o)}(x,t)$, we first integrate equation (3.3b) from $y_1^{(o)}$ to $y_2^{(o)}$ and obtain:

$$u_x^{(o)}(y_2^{(o)} - y_1^{(o)}) = u^{(1)}y_{ox} - v^{(1)}\Big]_{y_1^{(o)}}^{y_2^{(o)}} \quad . \tag{3.11}$$

Now, subtracting equation (3.3g) at $y = y_1^{(o)}$ from the same equation at $y_2^{(o)}$, and employing equations (3.3a) and (3.11), we have

$$y_{1t}^{(o)} + (u^{(o)}y_1^{(o)})_x = 0 \quad . \tag{3.12}$$

This equation for $y_1^{(o)}$ can also be solved by assuming $y_1^{(o)}(x,t) \equiv P(\tau,t)$. We have, after making this substitution:

$$P_t + Pu_x^{(o)} = 0 \quad . \tag{3.13}$$

This is a first order ordinary differential equation and can be solved immediately yielding for P or $y_1^{(o)}$:

$$y_1^{(o)}(x,t) = -y_2^{(o)}(x,t) = \frac{C(\tau)}{x_\tau^* + (t-\tau)A_\tau - A} \quad . \tag{3.14}$$

Here $C(\tau)$ is an arbitrary function, which may be related to the position of the wall of the nozzle (or to the nozzle width) at time τ.

We have now determined all the zero-order terms (and one first order term) in the solution. Collecting our results, we have in zero-order:

$$\begin{aligned} u &= A(\tau) + \ldots \\ v &= -(t-\tau) + W(\tau) + \ldots \\ p &= 0 + \sigma \cdot 0 + \ldots \\ y_o &= y^*(\tau) + (t-\tau)\,W(\tau) - \frac{(t-\tau)^2}{2} \\ y_1 &= -y_2 = C(\tau)\,[x_\tau^* + (t-\tau)A_\tau - A]^{-1} \end{aligned} \tag{3.15}$$

where τ is defined by

$$x = (t-\tau)\,A(\tau) + x^*(\tau) \quad .$$

The functions x^*, y^*, C, A, W are arbitrary. If the additional conditions on the flow are that it emerge from a nozzle, then x^* and y^* must be taken as the coordinates of the center of the nozzle, and C, A, W are chosen to fit the width and velocity of the jet at the nozzle, all at time τ. We see that, to this order, only the average velocity over the cross-section can be imposed as the boundary condition. This

is due to the asymptotic character of the approximation. In the higher order solution more details of the velocity at a cross-section can be imposed. We must also point out that the solution fails at points where $y_0(x,t)$ intersects itself.

The solution (3.15) resembles the flow of independent particles, each of which maintains a constant horizontal velocity and describes a parabolic trajectory. The pressure is zero (even to first order) but the width of the jet varies, showing a "bunching" effect. For example, the jet thickness (for fixed τ) increases, remains constant, or decreases according as A_τ is positive, zero, or negative. If gravity is neglected, the only change is the omission of the last term (one) in equation (3.3d) and consequently, the solution equations (3.15) still holds provided that we omit the first term in the expression for v and the last term in the expression for y_0.

If the flow is steady (independent of t) x^*, y^*, C, A, and W are constant. Then the solution equations (3.15) holds provided that we replace $t-\tau$ by xA^{-1} from equation (3.7). The result is

$$u = A + \dots$$

$$v = W - xA^{-1} + \dots$$

$$p = 0 + \sigma C + \dots \tag{3.16}$$

$$y_0 = y^* + WA^{-1}x - \frac{1}{2}A^{-2}x^2$$

$$y_1 = -y_2 = CA^{-1} + \dots \quad .$$

In this case the vertical width of the jet remains constant to this order. This solution does not apply for vertical jets, in which case A = 0. A somewhat different method must be used to treat this case, and it is described in Appendix I.

The zero order solutions obtained in this section are exactly the same as the results of the hydraulic theory. Thus the hydraulic theory yields a first approximation to the solution of the hydrodynamic problem, valid for thin jets.

4. First Order Solution for Steady Jets

We shall now proceed to the determination of the first order terms, but because of the complexity of the equations, we shall consider only the steady state solution.

From equations (3.3b, c and g) one easily finds that

$$
(4.1)\qquad
\begin{aligned}
u^1(x,y) &= u^1(x,y_1^o) + v_x^o(1+(\mu_{ox}^2)^2)^{-1}(y-y_1^o) \\
v^1(x,y) &= y_{ox}u^1(x,y) \quad .
\end{aligned}
$$

Equations (4.1) and the preceding expressions for p^1 and the zero order solutions satisfy equations (3.3b-g). To complete the determination of u^1 and v^1 and to find y^1, it is necessary to equate to zero coefficients of σ^2 in equations (2.3) and (2.4). From the second and third equations resulting from equations (2.4) we eliminate $p_y^{(2)}$ and, by substitutions from equation (4.1) we obtain an ordinary differential equation in u^1. This equation is easily solved and thus (from equation (4.1)) both u^1 and v^1 are found. Now employing these results in the equations resulting from the first and sixth of equations (2.4), we obtain y_i^1. This completes the determination of the first order solution.

The resulting solution, including both zero and first order term, is:

$$
\begin{aligned}
u &= A + \sigma(A^4D - A^3y)(A^4 + A^2W^2 - 2AWx + x^2)^{-1} + \dots \\
v &= W - A^{-1}x + \sigma(A^2D - Ay)(A^4 + A^2W^2 - 2AWx + x^2)^{-1}(AW - x) + \dots \\
(4.2)\qquad p &= 0 + \sigma\cdot 0 + \dots \\
y_o &= y^* + A^{-1}Wx - \tfrac{1}{2}A^{-2}x^2 \\
y_1 = -y_2 &= -CA^{-1} - \sigma CA^2D(A^4 + A^2W^2 - 2AWx + x^2)^{-1} + \dots \quad .
\end{aligned}
$$

Here D is a constant, which like the other constants, may be determined from the prescribed boundary values of the solution at some cross-section of the stream.

5. Unsteady Flow on a Rigid Surface (e.g. Hill or Spillway)

In the preceding problem the jet had two free streamlines. But in the flow over a rigid surface, such as a hill, spillway, river bed, etc. there is only one free streamline (except when the flow detaches itself from the surface). Such flows with one free streamline can be treated by the preceding method, with a slight modification. In fact, it was for the treatment of a problem with one free streamline that the method was originally employed - namely the problem of waves in shallow water. However, in that problem the "bottom" was assumed to be practically horizontal, and its slope small. In the present problem we wish to consider arbitrary "bottom" surfaces which may have large slopes. For such cases the original discussion of the shallow water theory does not apply.

We formulate the problem just as in section 2. However, the last two of equations (2.1) apply only at the single free surface $\bar{y} = \bar{y}_1(\bar{x},\bar{t})$. At the "bottom" surface $\bar{y} = \bar{y}_0(\bar{x})$ which is assumed to be known, we have the condition

$$\bar{y}_{o\bar{x}}\, \bar{u} = \bar{v} \quad \text{at} \quad \bar{y} = \bar{y}_0(\bar{x}) \quad . \tag{5.1}$$

The dimensionless variables in equations (2.2) and the equations (2.4) are the same. However, $\bar{y}_0(\bar{x}) = ay_0(x)$ represents the bottom surface instead of the center line as in the previous case. The derivative y_{ot} is also identically zero. Equation (5.1) becomes

$$uy_{ox} = v \quad \text{at} \quad y = 0 \quad . \tag{5.2}$$

The series representation of the solution, equation (2.5) and the equations (2.6) are the same as before. The method of solution is also similar to that of the previous case.

The zero-order solution (with p given to first order) is found to be:

$$u = \left[\frac{k(\tau) - 2y_o}{1 + y_{ox}^2}\right]^{1/2} + \ldots$$

$$v = \left[\frac{k(\tau) - 2y_o}{1 + y_{ox}^2}\right]^{1/2} + \ldots$$

(5.3) $$p = 0 + \sigma(y_1^{(0)} - y)(1 + y_{ox}^2 + y_{oxx}[k(\tau) - 2y_o])(1 + y_{ox}^2)^{-2} + \ldots$$

$$y_1 = N(\tau)\tau_x + \ldots = N(\tau)\left[\frac{1 + y_{ox}^2}{k(\tau) - 2y_o}\right]^{1/2}$$

$$\cdot\left\{-1 + \frac{k_\tau}{2}\int_{x^*}^{x}(1 + y_{ox}^2)^{1/2}(k(\tau) - 2y_o)^{-3/2}d\alpha\right.$$

$$\left. + x_\tau^*\left[\frac{1 + y_{ox}^2(x^*)}{k(\tau) - 2y_o(x^*)}\right]^{1/2}\right\} + \ldots$$

where τ is defined as a function of x and t by

(5.4) $$t - \tau = \int_{x^*(\tau)}^{x}\left[\frac{k(\tau) - 2y_o(\alpha)}{1 + y_{ox}^2(\alpha)}\right]^{-1/2} d\alpha \quad .$$

Here $k(\tau)$, $x^*(\tau)$ and $N(\tau)$ are arbitrary functions. The flow resembles a stream of individual particles sliding along the rigid surface. Then $k(\tau)$ and $x^*(\tau)$ represent the kinetic energy and x-coordinate at time τ of the particle which is at x at time t. The depth y_1 of the stream exhibits a "bunching" effect as in the previous problem. Even to this order mass and energy are conserved.

The pressure, given by the third of equations (5.3), is approximately hydrostatic where the bottom slope y_{ox} is small. However, if the slope is large, the variation of pressure with

depth is much less than what would be given by hydrostatics. If the pressure at some point on the bottom ($y = 0$) becomes zero, (and would be negative for neighboring points) then the stream detaches itself from the bottom at this point, and becomes a stream with two free streamlines, to which the previous solution applies. This will occur if

$$(5.5) \qquad 1 + y_{ox}^2 + y_{oxx}(k(\tau) - 2y_o) = 0 \quad .$$

Thus the stream detaches when the bottom curvature becomes too great. In fact the curvature at the point of detachment is (in the original variables)

$$(5.6) \qquad \bar{y}_{o\bar{x}\bar{x}}(1 + \bar{y}_{o\bar{x}}^2)^{-3/2} = -g\bar{u}^{-2}(1 + \bar{y}_{o\bar{x}}^2)^{-3/2} \quad .$$

This is the curvature of the trajectory of a freely falling particle with horizontal velocity $\bar{u}$ and trajectory slope $\bar{y}_{o\bar{x}}$. Thus we see that if the curvature of the bottom exceeds that of the path of a freely falling particle with the same horizontal velocity and slope as the stream, detachment occurs. (If the pressure is zero but not negative on the bottom, it is not necessary to speak of detachment since the "detached" stream would also have the bottom as its streamline.)

If the flow is steady (independent of t) the above solution (5.3) is valid with k, x^* and N constants. Equation (5.4) is then unnecessary, and the remaining discussion applies.

6. The Effect of Surface Tension on a Jet

In order to take account of surface tension $\bar{T}$, we need merely modify the fifth of (2.1), which now becomes

$$(2.1)^* \qquad \bar{p} = (-1)^i\, \bar{T}\, \bar{K}_i \qquad \text{at } \bar{y} = \bar{y}_i(\bar{x},\bar{t}) \;, \quad (i = 1,2) \quad .$$

The sign is determined by requiring that i = 1 corresponds to the upper surface, and $\bar{K}_i$ is the curvature of surface i. In

the introduction of dimensionless variables we include T, K_i defined by

$$\overline{T} = \rho gabT \quad , \quad \overline{K}_i = a^{-1}K_i \quad .$$

Then the fifth of (2.4) becomes

$$(2.4)^* \qquad p = (-1)^i \sigma TK_i \quad \text{at} \quad y = y_i(x,t) \quad .$$

The subsequent expressions given by (2.5) and the zero order equations (2.6) and their consequences (3.1), (3.2) remain unchanged. The first change occurs in (3.3) f), which becomes

$$(3.3)\ f)^* \qquad p^{(1)} = (-1)^i\ TK^{(o)} \quad \text{at} \quad y = y_i^{(o)} \quad .$$

As before (3.3) c) and (3.1) imply that $p_y^{(1)}$ is independent of y. Combining this fact with (3.3) f)* above, we find

$$(3.4)^* \qquad p^{(1)} = -TK^{(o)} \frac{y}{y_1^{(o)}} \quad .$$

This replaces the former (3.4). Using (3.4)*, (3.3) c) and (3.3) d) become

$$(3.5)^* \qquad u_t^{(o)} + u^{(o)}u_x^{(o)} = -\ TK^{(o)} \frac{y_{ox}}{y_1^{(o)}}$$

$$(3.6)^* \qquad v_t^{(o)} + u^{(o)}v_x^{(o)} = \frac{TK^{(o)}}{y_1^{(o)}} - 1$$

These two equations, along with (2.6) g) which is

$$(2.6)\ g) \qquad y_{ot} + u^{(o)}y_{ox} = v^{(o)}$$

provide three equations for the determination of $u^{(o)}$, $v^{(o)}$, y_o and $y_1^{(o)}$. The fourth equation is (3.12), as before

$$y_{1t}^{(o)} + (u^{(o)}y_1^{(o)})_x = 0 \quad . \tag{3.12}$$

Rewriting $(3.5)^*$, $(3.6)^*$, (2.6) g) and (3.12), omitting all indices and letting $h = y_1^{(o)}$ we have

$$u_t + uu_x = -\frac{Ty_x y_{xx}}{h(1+y_x^2)^{3/2}}$$

$$v_t + uv_x = \frac{Ty_{xx}}{h(1+y_x^2)^{3/2}} - 1$$

$$y_t + uy_x = v$$

$$h_t + (uh)_x = 0 \quad .$$

This is a set of four partial differential equations for the four functions u, v, y and h, which depend upon x and t. Exactly the same set of equations is obtained in the hydraulic theory, where the jet is treated as a stream of individual particles with mass proportional to the jet width, provided the force of surface tension is included as well as gravity. In that case however, the left hand sides may be thought of as total time derivatives.

We will now solve these equations when the jet is steady, i.e. independent of time. In this case the last equation above can be integrated, which introduces the constant flux c. Then the remaining equations can be simplified, yielding

$$uh = c$$

$$v = uy_x$$

$$cu_x = -\frac{Ty_x y_{xx}}{(1+y_x^2)^{3/2}}$$

$$c(uy_x)_x = \frac{Ty_{xx}}{(1+y_x^2)^{3/2}} - cu^{-1} \quad .$$

The third of these equations can be integrated from 0 to x, and yields

$$c(u - u(o)) = T[(1+y_x^2)^{-1/2} - (1+y_x^2(o))^{-1/2}] \quad .$$

Using this to eliminate u from the last preceding equation, we obtain a second order equation for y. Introducing $\alpha = cu(o) - T(1+y_x^2(o))^{-1/2}$ this becomes

$$\alpha y_{xx}[T(1+y_x^2)^{-1/2} + \alpha] = -c^2 \quad .$$

Integrating from 0 to x we have

$$\alpha T[\sinh^{-1} y_x - \sinh^{-1} y_x(o)] + c^2[y_x - y_x(o)] = -c^2 x \quad .$$

The last equation is a first order equation for y(x) involving several parameters. It seems that this equation must be integrated numerically, and it is therefore important to reduce the number of parameters. To this end we may set $y_x(o) = 0$, since the resulting curve will apply to any initial slope if read from that slope on. Next, we introduce the new variables $\frac{c^2}{\alpha^2} x = \xi$ and $\frac{c^2}{\alpha^2} y = \eta$. In terms of these variables the equation becomes

$$\eta_\xi + \frac{T}{\alpha} \sinh^{-1} \eta_\xi = -\xi \quad .$$

This equation involves only one parameter, $\frac{T}{\alpha} = [\frac{cu(o)}{T} - 1]^{-1}$, and is therefore convenient for numerical integration. When $\frac{T}{\alpha} = 0$, the above equation yields the expected parabola $\eta = -\frac{\xi^2}{2}$. In figure 1 graphs of the solution of the above equation are shown for various values of $\frac{T}{\alpha}$ corresponding to various values

of $\frac{cu(o)}{T}$. In the ξ, η variables the curves with $\frac{T}{\alpha} > 0$ lie above the parabola obtained for $\frac{T}{\alpha} = 0$. However in the x,y variables these curves lie below the parabola. Since this is the physical plane, it is important to recognize that in the physical plane the jets with surface tension lie below the parabola provided $\frac{T}{\alpha} > 0$, or $\frac{cu(o)}{T} > 1$.

It is interesting to notice that $\frac{T}{\alpha}$ becomes negative when $\frac{cu(o)}{T} < 1$, i.e. when the kinetic energy $\frac{1}{2} \cdot 2hu^2(o)$ becomes less than the surface energy T. In this case the above equation yields the result that the jet, although starting horizontally, will at first rise until it reaches a point at which the curvature becomes infinite and the theory consequently fails (see Figure 1). This surprising phenomenon can occur only in extremely thin and slow jets since the surface tension is very small for all ordinary liquids. In such slow jets, near the orifice or nozzle, the hydraulic theory may not apply, and therefore the effect might not occur. However, in a fluid with a large surface tension, in which this effect could be expected for a reasonable jet velocity and thickness, the hydraulic theory should apply and the effect should occur. When $\frac{cu(o)}{T} = 1$, the steady state equations have no solution, and therefore the hydraulic theory cannot apply in this special case.

Appendix I

Vertical, Steady Jets

We now consider a jet which flows vertically, and for simplicity, is steady. The method of sections 2-4 is not applicable in this case, and therefore we will describe an alternative method. The problem is formulated just as in section 2, but the free streamlines are designated by $\bar{x} = \bar{x}_i(\bar{y})$, $i = 1,2$. Equations (2.1) all remain valid except the last which becomes

$$\bar{V}\bar{x}_{i\bar{y}} = \bar{u} \quad \text{at} \quad \bar{x} = \bar{x}_i(\bar{y}) \quad (i = 1,2) \quad . \tag{A.1}$$

A suitable transformation to dimensionless variables is given by

$$\begin{aligned} \bar{x} &= ax & \bar{u} &= \sqrt{gb}\, u & \bar{x}_i &= ax_i \\ \bar{y} &= by & \bar{v} &= \sqrt{gb}\, v & \bar{p} &= gb\rho p \quad . \end{aligned} \tag{A.2}$$

We now proceed as in the previous case, but with $\sigma = \frac{a}{b}$. The result of determining the zero and first order terms, for a symmetrical jet ($x_1 = -x_2$) is:

$$\begin{aligned} u &= 0 - \sigma(k_1-2y)^{-1/2}x + \ldots \\ v &= -(k_1-2y)^{1/2} - \sigma k_3(k_1-2y)^{-1/2} + \ldots \\ p &= 0 + \sigma\cdot 0 + \sigma^2[k_2^2(k_1-2y)^{-2} - x^2(k_1-2y)^{-1}] + \ldots \end{aligned} \tag{A.3}$$

$$x_1 = -x_2 = -k_2(k_1-2y)^{-1/2} + \sigma[-k_4(k_1-2y)^{-1/2} + k_2k_3(k_1-2y)^{-3/2}] + \ldots$$

Here k_1, k_2, k_3, k_4 are constants.

By introducing the original variables, we obtain

$$\bar{u} = -\bar{x}\bar{g}(k_1-2\bar{y})^{-1/2} + \ldots$$

$$\bar{v} = -\bar{g}(k_1-2\bar{y})^{+1/2} + k_2k_3(k_1-2\bar{y})^{-3/2} + \ldots$$

(A.4)

$$\bar{x}_1 = -\bar{x}_2 = -(k_2+k_4)(k_1-2\bar{y})^{-1/2} + k_2k_3(k_1-2\bar{y})^{-3/2} + \ldots$$

$$\bar{p} = \rho g\ k_2^2(k_1-2\bar{y})^2 - \rho g\bar{x}^2(k_1-2\bar{y})^{-1} \quad .$$

In equations (A.4), k_1-k_4 are constants to be determined from the boundary conditions at some cross-section of the jet. The last form of the solution was given to show that the particular choice of a and b, or σ , does not affect the solution. This is also the case with our previous results.

The jet is symmetric about the vertical center line and its vertical velocity varies approximately like that of a freely falling body. Its width varies inversely as the vertical velocity, as we would expect from conservation of mass. A small horizontal velocity toward or away from the center of the jet accounts for its thinning (when falling) or broadening (when rising). (The positive square root applies to falling jets; the negative square root must be used for rising jets.)

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Steady Jets With Surface Tension
Falling due to Gravity

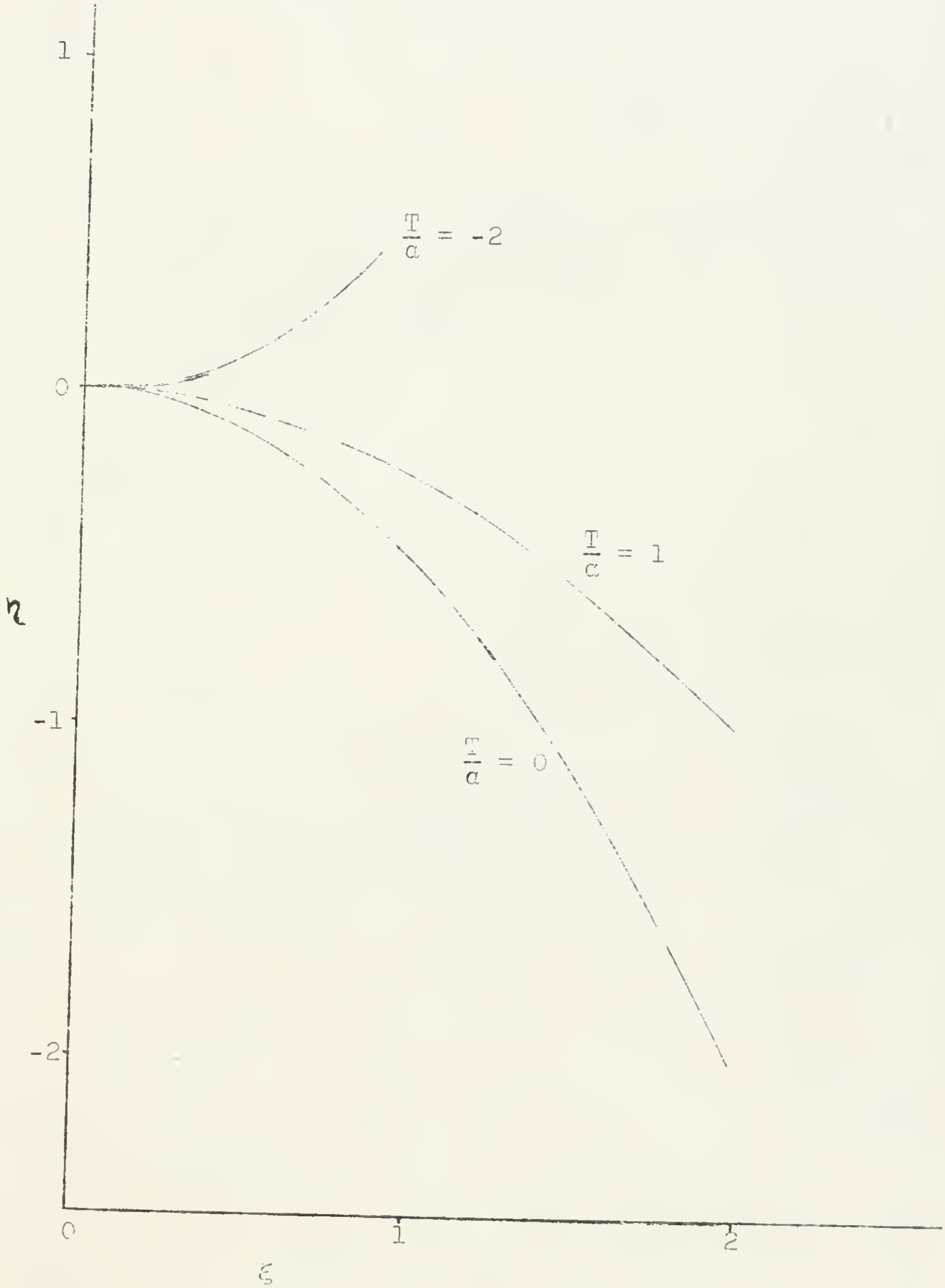

Figure 1